ELSEVIER Journal of Pure and Applied Algebra 127 (1998) 289–307

JOURNAL OF PURE AND APPLIED ALGEBRA

The construction of some Gorenstein ideals of codimension 4

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Communicated by F.W. Lawvere; received 23 May 1996; revised 25 September 1996

Abstract

We obtain the number and the degrees of the generators of the ideal of a k-configuration in $\mathbb{P}^3$ and so construct the Gorenstein ideal of codimension 4.

AMS Classification: Primary: 13D40; secondary 14M10

1. Introduction

Let k be an infinite field and let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s-distinct points in $\mathbb{P}^n$. We denote by $I(\mathbb{X})$ the defining ideal of $\mathbb{X}$ in the polynomial ring $R = k[x_0, \dots, x_n]$ and by A the homogeneous coordinate ring of $\mathbb{X}$, $A = \sum_{t=0}^{\infty} A_t$. The Hilbert function of $\mathbb{X}$ (or of A) is the function $\mathbf{H}: \mathbb{N} \to \mathbb{N}$ described by

$$\mathbf{H}(\mathbb{X}, t) = \mathbf{H}(A, t) = \dim_k A_t = \dim_k R_t - \dim_k I_t.$$

The first difference of the Hilbert function of $\mathbb{X}$ (or of A) is

$$\Delta\mathbf{H}(\mathbb{X}, t) = \begin{cases} 1 & \text{for } t = 0, \\ \mathbf{H}(\mathbb{X}, t) - \mathbf{H}(\mathbb{X}, t-1) & \text{for } t \geq 1. \end{cases}$$

We also denote by $\sigma(\mathbb{X})$ (or $\sigma(A)$) the least integer for which

$$\Delta\mathbf{H}(\mathbb{X}, \sigma) = 0 \quad \text{and} \quad \Delta\mathbf{H}(\mathbb{X}, \sigma - 1) \neq 0.$$

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PII S0022-4049(96)00179-X

In [5], we obtained the number and the degrees of the generators of an ideal of a k-configuration in $\mathbb{P}^2$ and the minimal graded free resolution of the ideal.

The aim of this paper is to obtain the number and the degrees of the generators of an ideal of a k-configuration in $\mathbb{P}^3$ and to construct a Gorenstein ideal of codimension 4 from them.

In Section 2, we recall the notion of a k-configuration in $\mathbb{P}^3$ and prove our main theorem (see Theorem 2.5).

In Section 3, we find the minimal graded free resolution of the ideal I of a k-configuration in $\mathbb{P}^3$ in a special case.

In Section 4, we introduce the notion of a weak k-configuration in $\mathbb{P}^3$ and construct Gorenstein ideals of codimension 4 using the results in Section 2. In [4], Geramita and Migliore constructed a Gorenstein ideal of codimension 3 using an arithmetically Cohen–Macaulay curve in $\mathbb{P}^3$ with a specific minimal free resolution, and the structure theorem of Buchsbaum and Eisenbud [2].

2. k-Configurations in $\mathbb{P}^3$

Roberts and Roitman [8] introduced the following definition:

Definition 2.1. A *k-configuration* is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

There exist integers $1 \le d_1 < \cdots < d_m$, and subsets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^m \mathbb{X}_i$;

(2) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \ldots, m$; and

(3) $\mathbb{L}_i$ $(1 < i \le m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the k-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \ldots, d_m)$.

Theorem 2.2 (Geramita et al. [5]). *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of type $(d_1, \ldots, d_m)$ and let I be the ideal of $\mathbb{X}$. Then $v(I) = m + 1$ and the minimal graded free resolution of I as an R-module is*

$$\begin{aligned} 0 &\to R(-(d_1 + m)) \oplus \cdots \oplus R(-(d_i + m - i + 1)) \oplus \cdots \oplus R(-(d_m + 1)) \\ &\to R(-m) \oplus R(-(d_1 + m - 1)) \oplus \cdots \oplus R(-(d_i + m - i)) \\ &\quad \oplus \cdots \oplus R(-d_m) \to I \to 0, \end{aligned}$$

where $v(I)$ is the number of the minimal generators of I.

Definition 2.3 (*Harima* [6]). A *k-configuration* in $\mathbb{P}^3$ is a finite set of points which satisfies the following conditions:

There exist subsets $\mathbb{X}_1, \ldots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct hyperplanes $\mathbb{H}_1, \ldots, \mathbb{H}_u$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^u \mathbb{X}_i$;

(2) $\mathbb{X}_i \subset \mathbb{H}_i$ for any $i = 1, \ldots, u$;

(3) $\mathbb{H}_i$ $(1 < i \le u)$ does not contain any points of $\mathbb{X}_j$ for any $j < i$; and

(4) $\mathbb{X}_i$ $(1 \le i \le u)$ is a k-configuration in $\mathbb{H}_i$ of type $(d_{i1},\dots,d_{im_i})$ with $d_{im_i} < m_{i+1}$ for every $1 \le i < u$.

In this case, the k-configuration in $\mathbb{P}^3$ is said to be of type

$$(d_{11},\dots,d_{1m_1};\dots;d_{u1},\dots,d_{um_u}).$$

For simplicity of notation, let (d_{ij}) denote the tuple of integers $(d_{11},\dots,d_{1m_1};\dots;d_{u1},\dots,d_{um_u})$ with $d_{im_i} < m_{i+1}$ for every $1 \le i < u$.

Remark 2.4. (1) All k-configurations in $\mathbb{P}^3$ of type (d_{ij}) have the same Hilbert function, which will be denoted by $\mathbf{H}^{(d_{ij})}$.

(2) Let $\mathbf{H} = \{b_t\}_{t\ge 0}$ be a zero-dimensional O-sequence with $b_1 = 4$. Applying the procedure of Theorem 4.1 in [3], we can get integers $(d_{11},\dots,d_{1m_1};\dots;d_{u1},\dots,d_{um_u})$ with $d_{im_i} < m_{i+1}$ for every $1 \le i < u$ such that

$$\mathbf{H} = \mathbf{H}^{(d_{ij})}.$$

Theorem 2.5. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11},\dots,d_{1m_1};\dots;d_{u1},\dots,d_{um_u})$ and let I be the ideal of $\mathbb{X}$. Then $v(I) = \sum_{i=1}^{u} m_i + u + 1$ and the degrees of the minimal generators of I are*

$$u,\ m_1+u-1,\ d_{11}+m_1+u-2,\ \dots,\ d_{1i}+m_1+u-i-1,\ \dots,\ d_{1m_1}+u-1,$$
$$\vdots$$
$$m_j+u-j,\ d_{j1}+m_j+u-j-1,\ \dots,\ d_{ji}+m_j+u-i-j,\ \dots,\ d_{jm_j}+u-j,$$
$$\vdots$$
$$m_u,\ d_{u1}+m_u-1,\ \dots,\ d_{ui}+m_u-i,\ \dots,\ d_{um_u}.$$

Proof. Since $\mathbb{X}$ is a k-configuration in $\mathbb{P}^3$ of type $(d_{11},\dots,d_{1m_1};\dots;d_{u1},\dots,d_{um_u})$, there exist subsets $\mathbb{X}_1,\dots,\mathbb{X}_u$ of $\mathbb{X}$ and distinct hyperplanes $\mathbb{H}_1,\dots,\mathbb{H}_u$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$;

(2) $\mathbb{X}_i \subset \mathbb{H}_i$ for every $i = 1,\dots,u$;

(3) $\mathbb{H}_i$ $(1 < i \le u)$ does not contain any points of $\mathbb{X}_j$ for any $j < i$, and;

(4) $\mathbb{X}_i$ $(1 \le i \le u)$ is a k-configuration in $\mathbb{P}^2$ of type $(d_{i1},\dots,d_{im_i})$ with $d_{im_i} < m_{i+1}$ for every $i < u$.

We shall prove the theorem by induction on u. Let $u = 1$. Then $\mathbb{X}$ is a k-configuration in $\mathbb{P}^2$ of type $(d_{11},\dots,d_{1m_1})$. Let $H_1 = I(\mathbb{H}_1)$, $S = R/(H_1)$ and $J_1 = (I + (H_1))/(H_1)$ $(= I/(H_1))$. Then J_1 is the ideal of a k-configuration in $\mathbb{P}^2$ of type $(d_{11},\dots,d_{1m_1})$. By Theorem 2.2, there exist $F_{10}, F_{11},\dots,F_{1m_1} \in I$ with degrees

$$\deg F_{10} = m_1,\quad \deg F_{11} = d_{11} + m_1 - 1,\quad \dots,\quad \deg F_{1m_1} = d_{1m_1}$$

such that $J_1 = \langle \overline{F}_{10}, \overline{F}_{11},\dots,\overline{F}_{1m_1}\rangle$. Hence

$$I = \langle H_1, F_{10}, F_{11},\dots,F_{1m_1}\rangle,$$

and this proves the theorem when $u = 1$.

Now suppose $u > 1$. Let $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$ and I' be the ideal of $\mathbb{Y}$. Then $\mathbb{Y}$ is a k-configuration in $\mathbb{P}^3$ of type $(d_{11},\dots,d_{1m_1};\dots;d_{(u-1)1},\dots,d_{(u-1)m_{u-1}})$. Hence $v(I') = \sum_{i=1}^{u-1} d_{im_i} + u$ and the degrees of the minimal generators of I' are

$$
\begin{aligned}
&u-1,\ m_1+(u-1)-1,\ d_{11}+m_1+(u-1)-2,\ \dots,\\
&d_{1i}+m_1+(u-1)-i-1,\ \dots,\ d_{1m_1}+(u-1)-1,\\
&\quad\vdots\\
&m_j+(u-1)-j,\ d_{j1}+m_j+(u-1)-j-1,\ \dots,\\
&d_{ji}+m_j+(u-1)-i-j,\ \dots,\ d_{jm_j}+(u-1)-j,\\
&\quad\vdots\\
&m_{u-1},\ d_{(u-1)1}+m_{u-1}-1,\ \dots,\ d_{(u-1)i}+m_{u-1}-i,\ \dots,\ d_{(u-1)m_{u-1}},
\end{aligned}
$$

by the induction hypothesis. Let $H_u = I(\mathbb{H}_u)$, $T = R/(H_u)$, and $J_2 = (I + (H_u))/(H_u)$. Then

$$\frac{I}{H_u\cdot[I:H_u]} = \frac{I}{(H_u)\cap I} \simeq \frac{I+(H_u)}{(H_u)} = J_2 \subset T.$$

Thus we have an exact sequence of graded modules

$$0 \to [I:H_u](-1) \xrightarrow{\times H_u} I \to \underset{\underset{\textstyle J_2}{\|}}{\frac{I+(H_u)}{(H_u)}} \to 0. \tag{2.1}$$

Let

$$
\begin{aligned}
&\mathbb{Y} = \{P_1,\dots,P_s\},\\
&\mathbb{X}_u = \{P_{s+1},\dots,P_{s+t}\},\\
&\wp_i = I(P_i),\quad \text{for every } i = 1,\dots,s+t.
\end{aligned}
$$

Since

$$[\wp_i : H_u] = \begin{cases} R & \text{if } H_u \in \wp_i,\\ \wp_i & \text{if } H_u \notin \wp_i,\end{cases}$$

we have, for every $i = 1,\dots,s+t$, the following:

$$[I:H_u] = \left[\bigcap_{i=1}^{s+t} \wp_i : H_u\right] = \bigcap_{i=1}^{s+t}[\wp_i : H_u] = \bigcap_{i=1}^{s}[\wp_i : H_u] = \bigcap_{i=1}^{s}\wp_i = I(\mathbb{Y}).$$

Thus we can rewrite the exact sequence (2.1) as

$$0 \to I(\mathbb{Y})(-1) \xrightarrow{\times H_u} I \to J_2 \to 0. \tag{2.2}$$

It follows from (2.2) that

$$\mathbf{H}(T/J_2,t) = \begin{cases} 1 & \text{for } t=0, \\ \mathbf{H}(R/I,t) - \mathbf{H}(\mathbb{Y},t-1) & \text{for } t \geq 1, \end{cases}$$
$$= \mathbf{H}(\mathbb{X}_u,t),$$

which implies J_2 is a saturated ideal, i.e., $I+(H_u)=I(\mathbb{X}_u)$.

By Theorem 2.2, there exist $F_{u0},F_{u1},\dots,F_{um_u} \in I$ with degrees

$$\deg F_{u0} = m_u, \quad \deg F_{u1} = d_{u1} + m_u - 1,\dots, \quad \deg F_{um_u} = d_{u\,m_u}$$

such that $\overline{F}_{u0},\overline{F}_{u1},\dots,\overline{F}_{um_u}$ are the minimal generators of J_2. Let $\{F'_{ij}\}$ be the minimal generators of $I(\mathbb{Y})$ and $\{F_{ij}\}=\{F'_{ij}H_u\} \cup \{F_{u0},F_{u1},\dots,F_{um_u}\}$. □

Claim. $I=\langle\{F_{ij}\}\rangle$.

Proof of claim. Clearly, $\langle\{F_{ij}\rangle \subseteq I$. Conversely, for every $F \in I$, $\overline{F} \in J_2$. Hence

$$F = F_{u0}N_0 + F_{u1}N_1 + \cdots + F_{um_u}N_{m_u} + H_uK$$

for some $N_0,N_1,\dots,N_{m_u}$, $K \in R$. Since $K \in [I:H_u]=I(\mathbb{Y})$,

$$K = \sum F'_{ij}M_{ij}$$

for some $M_{ij} \in R$. Hence

$$\begin{aligned} F &= F_{u0}N_0 + F_{u1}N_1 + \cdots + F_{um_u}N_{m_u} + H_uK \\ &= F_{u0}N_0 + F_{u1}N_1 + \cdots + F_{um_u}N_{m_u} + H_u \sum F'_{ij}M_{ij} \\ &= F_{u0}N_0 + F_{u1}N_1 + \cdots + F_{um_u}N_{m_u} + \sum (F'_{ij}H_u)M_{ij} \\ &\in \langle\{F_{ij}\}\rangle. \end{aligned}$$

Hence we are done. □

3. A graded free resolution of the ideal of a k-configuration in $\mathbb{P}^3$

From Theorem 2.2, we can always get a minimal graded free resolution of the ideal of a k-configuration in $\mathbb{P}^2$. But it is not easy to get a minimal graded free resolution of the ideal of a k-configuration in $\mathbb{P}^3$.

Let $\mathscr{S}(R/I)$ denote the socle elements of R/I when $\dim R/I=0$ and $I \neq m = (x,y,z,w)$ where $R=k[x,y,z,w]$. From the minimal graded free resolution in Theorem 2.2, we get the following Lemma.

Lemma 3.1. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of type $(d_1,\dots,d_m)$ and I be the ideal of $\mathbb{X}$. Let L be a general linear form of R/I. Then there exist $A_1,\dots,A_i,\dots,A_m \in R$ such that*

$$\mathscr{S}(R/(L,I)) = (\bar{A}_1,\dots,\bar{A}_i,\dots,\bar{A}_m)$$

where

$$\deg A_1 = d_1 + m - 2, \quad \dots, \quad \deg A_i = d_i + m - i - 1, \quad \dots, \quad \deg A_m = d_m - 1.$$

Lemma 3.2. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_1,\dots,d_m)$, I be the ideal of $\mathbb{X}$, $\mathbb{H}$ be a hyperplane which contains $\mathbb{X}$, and $(H) = I(\mathbb{H})$. Let L be a general linear form of R/I, $S = R/(H)$, and $J = I/(H) \subset S$. Then for some $A \in R$, $\bar{A}$ is a non-zero socle element of $R/(L,I)$ if and only if $\bar{A}$ is a non-zero socle element of $S/(\bar{L},J)$. In particular,*

$$\dim_k \mathscr{S}(R/(L,I)) = \dim_k \mathscr{S}(S/(\bar{L},J)).$$

Proof. Clearly, A is not zero in $R/(L,I)$ if and only of $\bar{A}$ is not zero in $S/(\bar{L},J)$ and

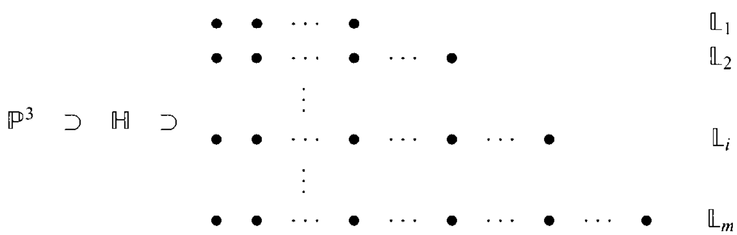
$$A\mathfrak{m} \subset (L,I) \Leftrightarrow \bar{A}\bar{\mathfrak{m}} = (A\mathfrak{m} + (H))/(H) \subset (L,I)/(H) = (\bar{L},J),$$

i.e.,

$$A \in \mathscr{S}(R/(L,I)) \Leftrightarrow \bar{A} \in \mathscr{S}(S/(\bar{L},J)).$$

Hence we are done. □

Lemma 3.3. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_1,\dots,d_m)$ and I be the ideal of $\mathbb{X}$.*



Let L be a general linear form of R/I. Then there exist $A_1,\dots,A_i,\dots,A_m \in R$ such that

$$\mathscr{S}(R/(L,I)) = (\bar{A}_1,\dots,\bar{A}_i,\dots,\bar{A}_m)$$

where

$$\deg A_1 = d_1 + m - 2, \quad \dots, \quad \deg A_i = d_i + m - i - 1, \quad \dots, \quad \deg A_m = d_m - 1.$$

Proof. We shall prove this by induction on m. Let $m = 1$. Then $\mathbb{X}$ is a complete intersection of type $(1,1,d_1)$. Hence there exists a socle element $\bar{A}_1$ of $R/(L,I)$ with degree $d_1 - 1$.

Now assume $m > 1$. Let $\mathbb{L}_1, \ldots, \mathbb{L}_m$ ($\neq \mathbb{H}$) be the distinct hyperplanes such that $\mathbb{L}_i$ contains the ith d_i-points of $\mathbb{X}$ for every $i = 1, \ldots, m$.

Let $\mathbb{Y} = \bigcup_{i=1}^{m-1} \mathbb{X}_i$ where $\mathbb{X}_i = \mathbb{X} \cap \mathbb{L}_i$ for every $i = 1, \ldots, m$. Then $\mathbb{Y}$ is a k-configuration in $\mathbb{P}^3$ of type $(d_1, \ldots, d_{m-1})$. Hence there exist $A'_1, \ldots, A'_i, \ldots, A'_{m-1} \in R$ such that

$$\mathcal{S}(R/(L, I(\mathbb{Y}))) = (\bar{A}'_1, \ldots, \bar{A}'_i, \ldots, \bar{A}'_{m-1})$$

with degrees $d_1 + (m-1) - 2, \ldots, d_i + (m-1) - i - 1, \ldots, d_{m-1} - 1$ by the induction hypothesis.

By Theorem 2.5, there exist minimal generators

$$H,\ F'_0,\ F'_1,\ \ldots,\ F'_i,\ \ldots,\ F'_{m-1}$$

of $I(\mathbb{Y})$ with degrees

$$\deg F'_0 = m - 1, \quad \deg F'_1 = d_1 + (m-1) - 1, \quad \ldots,$$
$$\deg F'_i = d_i + (m-1) - i, \quad \ldots, \quad \deg F'_{m-1} = d_{m-1}.$$

Let $(L_m) = I(\mathbb{L}_m)$ and $F_i = F'_i L_m$ for every $i = 0, \ldots, m-1$. Then

$$I = \langle H, F_0, F_1, \ldots, F_i, \ldots, F_m \rangle$$

for some $F_m \in I$ with degree d_m by Theorem 2.2 and 2.5.

Claim. $\bar{A}_1, \ldots, \bar{A}_{m-1}$ are socle elements of $R/(L, I)$ where $A_i = A'_i L_m$ for every $i = 1, \ldots, m-1$.

Proof of claim. Assume $\bar{A}_i = \bar{0}$ in $R/(L, I)$ for some $i = 1, \ldots, m-1$. Then $A_i \in (L, I) = \langle L, H, F_0, F_1, \ldots, F_m \rangle$. Since $\deg A_i = d_i + m - i - 1 < d_i + m - i = \deg F_i$, $A_i \in \langle L, H, F_0, \ldots, F_{i-1} \rangle$, there exist $\alpha, \gamma, \beta_0, \ldots, \beta_{i-1}$ such that

$$\begin{aligned} A_i &= \alpha L + \gamma H + \beta_0 F_0 + \cdots + \beta_{i-1} F_{i-1} \\ &\Rightarrow (A'_i - (\beta_0 F'_0 + \cdots + \beta_{i-1} F'_{i-1})) L_m \in \langle L, H \rangle. \end{aligned}$$

Hence

$$A'_i - (\beta_0 F'_0 + \cdots + \beta_{i-1} F'_{i-1}) \in \langle L, H \rangle$$

since L, H, L_m are a regular sequence in R. Thus

$$A'_i \in \langle L, H, F'_0, \ldots, F'_{i-1} \rangle \subset (L, I(\mathbb{Y})),$$

a contradiction. Hence $\bar{A}_i \neq \bar{0}$ in $R/(L, I)$ for all $i = 1, \ldots, m-1$. Moreover,

$$A_i \mathfrak{m} = (A'_i L_m)\mathfrak{m} = L_m (A'_i \mathfrak{m}) \subset L_m (L, I(\mathbb{Y})) = (L_m L, L_m I(\mathbb{Y})) \subset (L, I),$$

which implies that $\bar{A}_i$ is a non-zero socle element of $R/(L, I)$ with degree $d_i + m - i - 1$ for every $i = 1, \ldots, m-1$. Since $\bar{A}'_1, \ldots, \bar{A}'_{m-1}$ are linear independent over k, $\bar{A}_1, \ldots, \bar{A}_i, \ldots, \bar{A}_{m-1}$ are also linear independent over k.

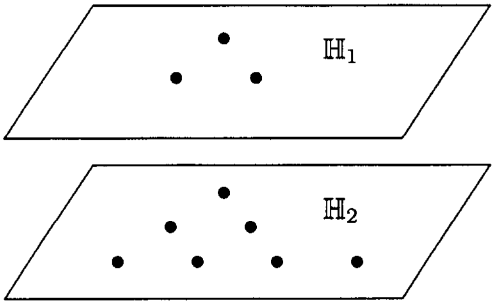


Fig. 1. k-configuration in $\mathbb{P}^3$ of type $(1,2;1,2,4)$.

Since

$$\begin{aligned}\Delta\mathbf{H}(\mathbb{X},d_m-1) &= \Delta[\mathbf{H}(\mathbb{X}_m,d_m-1)+\mathbf{H}(\mathbb{Y},d_m-2)] \\ &= \Delta\mathbf{H}(\mathbb{X}_m,d_m-1)+\Delta\mathbf{H}(\mathbb{Y},d_m-2) \\ &= 1+\Delta\mathbf{H}(\mathbb{Y},d_m-2)\end{aligned}$$

and $\Delta\mathbf{H}(\mathbb{X},d_m)=0$, we get one more socle element $\bar{A}_m$ of $R/(L,I)$ with degree d_m-1 which is not contained in $(\bar{A}_1,\ldots,\bar{A}_i,\ldots,\bar{A}_{m-1})$. Hence

$$\mathscr{S}(R/(L,I))=(\bar{A}_1,\ldots,\bar{A}_i,\ldots,\bar{A}_m)$$

with degrees

$$d_1+m-2,\ \ldots,\ d_i+m-i-1,\ \ldots,\ d_m-1$$

by Lemma 3.3 and Theorem 2.2, and we are done. □

From Lemma 3.3, we get the following theorem.

Theorem 3.4. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_1,\ldots,d_m)$ and let I be the ideal of $\mathbb{X}$. Then the minimal graded free resolution of I as an R-module is*

$$\begin{aligned}0 &\to R(-(d_1+m+1))\oplus\cdots\oplus R(-(d_m+2)) \\ &\to R(-(m+1))\oplus R^2(-(d_1+m))\oplus\cdots\oplus R^2(-(d_m+1)) \\ &\to R(-1)\oplus R(-m)\oplus R(-(d_1+m-1)\oplus\cdots\oplus R(-d_m) \\ &\to I\to 0.\end{aligned}$$

Example 3.5 (*Macaulay, see Bayer and Stillman* [1]). Let $\mathbb{X}=\{(0,0,1,1),(1,0,1,1),(1,1,1,1),(0,0,0,1),(1,0,0,1),(1,1,0,1),(2,0,0,1),(2,1,0,1)\ (2,2,0,1),(2,3,0,1)\}$ (see Fig. 1). Then $\mathbb{X}$ is a k-configuration in $\mathbb{P}^3$ of type $(1,2;1,2,4)$.

A computation, using Macaulay, gives that for this example, the Betti numbers in a minimal free resolution of the ideal of $\mathbb{X}$ are:

total:	1	8	12	5
0:	1	—	—	—
1:	—	1	—	—
2:	—	6	10	4
3:	—	1	2	1

Notice that these numbers are precisely those given in (3.1). This is not an isolated example. We have made many calculations (using Macaulay) and have always found the Betti numbers in a minimal free resolution of the ideal of a k-configuration in $\mathbb{P}^3$ are those given (3.1).

Hence, it seems reasonable to conjecture:

Conjecture 3.6. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11},\dots,d_{1m_1};\dots;d_{u1},\dots,d_{um_u})$ and let I be the ideal of $\mathbb{X}$. Then a graded minimal free resolution of I as an R-module is*

$$0 \to \bigoplus_{i=1}^{m_1+\cdots+m_u} R(-c_i) \to \bigoplus_{j=1}^{2(m_1+\cdots+m_u)+u} R(-b_j) \to \bigoplus_{k=1}^{m_1+\cdots+m_u+u+1} R(-a_k) \to I \to 0 \tag{3.1}$$

where

$$\begin{aligned}
&\bigoplus_{k=1}^{m_1+\cdots+m_u+u+1} R(-a_k)\\
&= R(-u)\oplus R(-(m_1+u-1))\oplus R(-(d_{11}+m_1+u-2))\oplus\cdots\\
&\quad \oplus R(-(d_{1i}+m_1-i+u-1))\oplus\cdots\oplus R(-(d_{1m_1}+u-1))\\
&\quad \oplus R(-(m_2+u-2))\oplus R(-(d_{21}+m_2+u-3))\oplus\cdots\\
&\quad \oplus R(-(d_{2i}+m_2-i+u-2))\oplus\cdots\oplus R(-(d_{2m_2}+u-2))\oplus\\
&\qquad\qquad \vdots\\
&\quad \oplus R(-(m_j+u-j))\oplus R(-(d_{j1}+m_j+u-j-1))\oplus\cdots\\
&\quad \oplus R(-(d_{ji}+m_j-i+u-j))\oplus\cdots\oplus R(-(d_{jm_j}+u-j))\oplus\\
&\qquad\qquad \vdots\\
&\quad \oplus R(-m_u)\oplus R(-(d_{u1}+m_u-1))\oplus\cdots\oplus R(-(d_{ui}+m_u-i))\\
&\quad \oplus\cdots\oplus R(-d_{um_u}),
\end{aligned}$$

$$\bigoplus_{j=1}^{2(m_1+\cdots+m_u)+u} R(-b_j)$$

$$\begin{aligned}
&= R(-(m_1+u))\oplus R^2(-(d_{11}+m_1+u-1))\oplus\cdots\oplus R^2(-(d_{1i}+m_1-i+u))\\
&\quad\oplus\cdots\oplus R^2(-(d_{1m_1}+u))\oplus R(-(m_2+u-1))\oplus R^2(-(d_{21}+m_2+u-2))\\
&\quad\oplus\cdots\oplus R^2(-(d_{2i}+m_2-i+u-1))\oplus\cdots\oplus R^2(-(d_{2m_2}+u-1))\oplus\\
&\quad\vdots\\
&\quad\oplus R(-(m_j+u-j+1))\oplus R^2(-(d_{j1}+m_j+u-j))\oplus\cdots\\
&\quad\oplus R^2(-(d_{ji}+m_j-i+u-j+1))\oplus\cdots\oplus R^2(-(d_{jm_j}+u-j+1))\oplus\\
&\quad\vdots\\
&\quad\oplus R(-(m_u+1))\oplus R^2(-(d_{u1}+m_u))\oplus\cdots\oplus R^2(-(d_{ui}+m_u-i+1))\\
&\quad\oplus\cdots\oplus R^2(-(d_{um_u}+1)),
\end{aligned}$$

$$\bigoplus_{i=1}^{m_1+\cdots+m_u} R(-c_i)$$

$$\begin{aligned}
&= R(-(d_{11}+m_1+u))\oplus\cdots\oplus R(-(d_{1i}+m_1-i+u+1))\oplus\cdots\\
&\quad\oplus R(-(d_{1m_1}+u+1))\oplus R(-(d_{21}+m_2+u-1))\oplus\cdots\\
&\quad\oplus R(-(d_{2i}+m_2-i+u))\oplus\cdots\oplus R(-(d_{2m_2}+u))\oplus\\
&\quad\vdots\\
&\quad\oplus R(-(d_{j1}+m_j+u-j+1))\oplus\cdots\oplus R(-(d_{ji}+m_j-i+u-j+2))\\
&\quad\oplus\cdots\oplus R(-(d_{jm_j}+u-j+2))\oplus\\
&\quad\vdots\\
&\quad\oplus R(-(d_{u1}+m_u+1))\oplus\cdots\oplus R(-(d_{ui}+m_u-i+2))\oplus\cdots\\
&\quad\oplus R(-(d_{um_u}+2)).
\end{aligned}$$

4. The construction of some Gorenstein ideals of codimension 4

In this section, we shall construct some Gorenstein ideals of codimension 4 using k-configurations in $\mathbb{P}^3$ and find the degrees of the minimal generators of these ideals.

Definition 4.1 (*Geramita et al.* [5]). A *weak k-configuration* in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

There exist integers $1 \le d_1 \le \cdots \le d_m$, subsets $\mathbb{X}_1, \dots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \dots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that:

(1) $i \le d_i$ for each $i = 1, \dots, m$;

(2) $\mathbb{X} = \bigcup_{i=1}^m \mathbb{X}_i$;

(3) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \dots, m$; and

(4) $\mathbb{L}_i$ $(1 < i \le m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the weak k-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \dots, d_m)$.

Theorem 4.2 (Geramita et al. [5]). *Let $\mathbb{X}$ be a weak k-configuration in $\mathbb{P}^2$ of type $(d_1, \dots, d_m, \dots, d_{m+\ell})$ where $d_1 < \cdots < d_m = \cdots = d_{m+\ell}$ and $\ell \ge 1$. Let I be the ideal of $\mathbb{X}$. If $\mathbb{X}$ is a subset of complete intersection in $\mathbb{P}^2$ of type $(m+\ell, d_m)$, then $\nu(I) = m+1$ and the minimal free resolution of I, as an R-module, is*

$$
\begin{aligned}
0 &\to R(-(d_1+m+\ell)) \oplus \cdots \oplus R(-(d_i+m+\ell-i+1)) \oplus \cdots \\
&\quad \oplus R(-(d_{m-1}+\ell+2)) \oplus R(-(d_m+\ell+1)) \\
&\to R(-(m+\ell)) \oplus R(-(d_1+m+\ell-1)) \oplus \cdots \\
&\quad \oplus R(-(d_i+m+\ell-i)) \oplus \cdots \oplus R(-(d_{m-1}+\ell+1)) \oplus R(-d_m) \\
&\to I \to 0.
\end{aligned}
$$

Definition 4.3. A *weak k-configuration* in $\mathbb{P}^3$ is a finite set of points which satisfies the following conditions:

There exist subsets $\mathbb{X}_1, \dots, \mathbb{X}_u$ of $\mathbb{X}$ and distinct hyperplanes $\mathbb{H}_1, \dots, \mathbb{H}_u$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^u \mathbb{X}_i$;

(2) $\mathbb{X}_i \subset \mathbb{H}_i$ for any $i = 1, \dots, u$;

(3) $\mathbb{H}_i$ $(1 < i \le u)$ does not contain any points of $\mathbb{X}_j$ for any $j < i$; and

(4) $\mathbb{X}_i$ $(1 \le i \le u)$ is a weak k-configuration in $\mathbb{H}_i$ of type $(d_{i1}, \dots, d_{im_i})$.

In this case, the weak k-configuration in $\mathbb{P}^3$ is said to be of type

$$(d_{11}, \dots, d_{1m_1}; \dots; d_{u1}, \dots, d_{um_u}).$$

From the Theorem 4.2, we obtain the following theorem.

Theorem 4.4. *Let $\mathbb{X}$ be a weak k-configuration in $\mathbb{P}^3$ of type $(d_1, \dots, d_m, \dots, d_{m+\ell})$ where $d_1 < \cdots < d_m = \cdots = d_{m+\ell}$ and $\ell \ge 1$. Let I be the ideal of $\mathbb{X}$. If $\mathbb{X}$ is a subset of complete intersection in $\mathbb{P}^3$ of type $(1, m+\ell, d_m)$, then $\nu(I) = m+2$ and the degrees of the minimal generators of I are*

$$1,\ m+\ell,\ d_1+m+\ell-1, \dots,\ d_i+m+\ell-i, \dots,\ d_{m-1}+\ell+1,\ d_m.$$

Definition 4.5 (*Harima* [6]). A finite complete intersection set of points $\mathbb{Z}$ in $\mathbb{P}^n$ is said to be a *basic configuration* in $\mathbb{P}^n$ if there exist integers $r_1, \dots, r_n$ and distinct hyperplanes $\mathbb{L}_{ij}(1 \le i \le n, 1 \le j \le r_i)$ such that

$$\mathbb{Z} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_n \text{ as schemes, } \quad \text{where } \mathbb{H}_i = \mathbb{L}_{i1} \cup \cdots \cup \mathbb{L}_{ir_i}.$$

In this case $\mathbb{Z}$ is said to be of type $(r_1, \ldots, r_n)$.

Remark 4.6. Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^3$ of type (u, α, β) $(u \le \alpha < \beta)$. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i \subset \mathbb{Z}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$ where $\mathbb{X}_i$ is a k-configuration in $\mathbb{P}^2$ of type $(d_{i1}, \ldots, d_{im_i})$. Let $m_u < \alpha$ and $d_{um_u} < \beta$. Assume $\mathbb{Z}_i \subset \mathbb{Z}$ is a basic configuration in $\mathbb{P}^3$ of type $(1, \alpha, \beta)$ such that $\mathbb{X}_i \subset \mathbb{Z}_i$ and $\mathbb{Y}_i = \mathbb{Z}_i - \mathbb{X}_i$ is a weak k-configuration $\mathbb{P}^3$ of type $(\beta - d_{im_i}, \ldots, \beta - d_{i1}, \beta, \ldots, \beta)$ for every $i = 1, \ldots, u$. Let $\mathbb{Y} = \bigcup_{i=1}^{u} \mathbb{Y}_i$. Then $\mathbb{Y}$ is a weak k-configuration in $\mathbb{P}^3$.

Moreover,

$$\Delta\mathbf{H}(\mathbb{Z}, t) = \Delta\mathbf{H}(\mathbb{X}, t) + \Delta\mathbf{H}(\mathbb{Y}, \sigma - 1 - t),$$

where $\sigma = \sigma(\mathbb{X}) = u + \alpha + \beta - 2$.

Similarly,

$$\Delta\mathbf{H}(\mathbb{Z}_u, t) = \Delta\mathbf{H}(\mathbb{X}_u, t) + \Delta\mathbf{H}(\mathbb{Y}_u, \sigma' - 1 - t),$$

$$\Delta\mathbf{H}(\mathbb{Z}', t - 1) = \Delta\mathbf{H}(\mathbb{X}', t - 1) + \Delta\mathbf{H}(\mathbb{Y}', \sigma - 1 - t),$$

where $\mathbb{Z}' = \bigcup_{i=1}^{u-1} \mathbb{Z}_i$, $\mathbb{X}' = \bigcup_{i=1}^{u-1} \mathbb{X}_i$, $\mathbb{Y}' = \bigcup_{i=1}^{u-1} \mathbb{Y}_i$, and $\sigma' = \alpha + \beta - 1$. Hence

$$\Delta\mathbf{H}(\mathbb{Y}, \sigma - 1 - t) = \Delta\mathbf{H}(\mathbb{Y}_u, \sigma' - 1 - t) + \Delta\mathbf{H}(\mathbb{Y}', \sigma - 1 - t).$$

Let $s = \sigma - 1 - t$. Since $\sigma' - \sigma = -u + 1$,

$$\Delta\mathbf{H}(\mathbb{Y}, s) = \Delta\mathbf{H}(\mathbb{Y}_u, s - u + 1) + \Delta\mathbf{H}(\mathbb{Y}', s).$$

Hence we obtain the following Lemma.

Lemma 4.7. *Let $\mathbb{Y}$, $\mathbb{Y}_u$, and $\mathbb{Y}'$ be as in Remark 4.6. Then*

$$\begin{aligned} &\Delta\mathbf{H}(\mathbb{Y}, s) = \Delta\mathbf{H}(\mathbb{Y}_u, s - (u - 1)) + \Delta\mathbf{H}(\mathbb{Y}', s), \\ &\quad \text{i.e.,} \\ &\mathbf{H}(\mathbb{Y}, s) = \mathbf{H}(\mathbb{Y}_u, s - (u - 1)) + \mathbf{H}(\mathbb{Y}', s) \end{aligned} \tag{4.1}$$

for every $s \ge 0$.

Remark 4.8. Let $\mathbb{Y}$ and $\mathbb{Y}_i$ be as in Remark 4.6 and let $\mathbb{Y}'' = \bigcup_{i=2}^{u} \mathbb{Y}_i$. Then, from (4.1),

$$\mathbf{H}(\mathbb{Y}, t) = \mathbf{H}(\mathbb{Y}_1, t) + \mathbf{H}(\mathbb{Y}'', t - 1). \tag{4.2}$$

Theorem 4.9. *Let $\mathbb{Y}$ be as in Remark 4.6. Let $J = I(\mathbb{Y})$. Then $\nu(J) = \sum_{i=1}^{u} m_i + 3$ and the degrees of the minimal generators of J are*:

$$\begin{array}{l} \beta - d_{1m_1} + \alpha - 1, \ \ldots, \ \beta - d_{11} + \alpha - m_1, \\ \beta - d_{2m_2} + \alpha, \ \ldots, \ \beta - d_{21} + \alpha - m_2 + 1, \\ \vdots \\ u, \ \alpha, \ \beta - d_{um_u} + \alpha + u - 2, \ \ldots, \ \beta - d_{u1} + \alpha - m_u + u - 1, \ \beta. \end{array}$$

Proof. Let $\mathbb{Y}_i$, $\mathbb{Z}$, and $\mathbb{Z}_i$ be as in Remark 4.6. Set $\mathbb{H}_i$ the hyperplane which contains $\mathbb{Z}_i$ and $H_i = I(\mathbb{H}_i)$. We shall prove the theorem by induction on u. If $u = 1$, then we are done by Theorem 4.4.

Now assume $u > 1$. Let $\mathbb{Y}''$ be as in Remark 4.8. Then, by the induction hypothesis, there exist $\sum_{i=2}^{u} m_i + 3$ minimal generators of $I(\mathbb{Y}'')$

$$\begin{array}{l} F'_{21}, \ \ldots, \ F'_{2m_2}, \\ \vdots \\ H_2 \ldots H_u, \ F'_{u0}, \ F'_{u1}, \ \ldots, \ F'_{um_u}, \ F'_{um_u+1}, \end{array}$$

with degrees

$$\begin{array}{l} \beta - d_{2m_2} + \alpha - 1, \ \ldots, \ \beta - d_{21} + \alpha - m_2, \\ \vdots \\ u - 1, \ \alpha, \ \beta - d_{um_u} + \alpha + u - 3, \ \ldots, \ \beta - d_{u1} + \alpha - m_u + u - 2, \ \beta, \end{array}$$

respectively, where $F'_{u0} = g$ and $F'_{um_u+1} = h$.

Let $S = R/(H_1)$ and $J' = \frac{J+(H_1)}{(H_1)}$. Then

$$\frac{J}{H_1 \cdot [J : H_1]} = \frac{J}{(H_1) \cap J} \simeq \frac{J + (H_1)}{(H_1)} = J' \subset S.$$

Thus we have an exact sequence of graded modules

$$0 \to [J : H_1](-1) \xrightarrow{\times H_1} J \to \underset{\substack{\| \\ J'}}{\frac{J + (H_1)}{(H_1)}} \to 0. \tag{4.3}$$

Since $[J : H_1] = I(\mathbb{Y}'')$, we can rewrite the exact sequence (4.3) as

$$0 \to I(\mathbb{Y}'')(-1) \xrightarrow{\times H_1} J \to J' \to 0. \tag{4.4}$$

It follows from (4.4) and (4.2) that

$$\begin{aligned} \mathbf{H}(S/J', t) &= \begin{cases} 1 & \text{for } t = 0 \\ \mathbf{H}(R/J, t) - \mathbf{H}(\mathbb{Y}'', t - 1) & \text{for } t \geq 1, \end{cases} \\ &= \mathbf{H}(\mathbb{Y}_1, t), \end{aligned}$$

which implies J' is a saturated ideal, i.e., $J + (H_1) = I(\mathbb{Y}_1)$.

By Theorem 4.4, there exist $F_{10}, F_{11}, \ldots, F_{1m_1}, F_{1m_1+1} \in J$ with degrees

$$\deg F_{10} = \alpha, \quad \deg F_{11} = \beta - d_{1m_1} + \alpha - 1, \quad \ldots,$$

$$\deg F_{1m_1} = \beta - d_{11} + \alpha - m_1, \quad \deg F_{1m_1+1} = \beta$$

such that $\overline{F_{10}}, \overline{F_{11}}, \ldots, \overline{F_{1m_1}}, \overline{F_{1m_1+1}}$ are the minimal generators of J'. Moreover, $F_{10} = g$ and $F_{1m_1+1} = h$. Let $\{F'_{ij}\}$ be the minimal generators of $I(\mathbb{Y}'')$ and $\{F_{ij}\} = \{F'_{ij}H_1\} \cup \{F_{10}, F_{11}, \ldots, F_{1m_1}, F_{1m_1+1}\}$.

Claim. $J = \langle \{F_{ij}\} \rangle$.

Proof of claim. Clearly, $\langle \{F_{ij}\} \rangle \subseteq J$. Conversely, for every $F \in J$, $\overline{F} \in J'$. Hence

$$F = F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + H_1K$$

for some $N_0, N_1, \ldots, N_{m_1}, N_{m_1+1}, K \in R$. Since $K \in [J : H_1] = I(\mathbb{Y}'')$,

$$K = \sum F'_{ij}M_{ij}$$

for some $M_{ij} \in R$. Hence

$$\begin{aligned} F &= F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + H_1K \\ &= F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + H_1 \sum F'_{ij}M_{ij} \\ &= F_{10}N_0 + F_{11}N_1 + \cdots + F_{1m_1}N_{m_1} + F_{1m_1+1}N_{m_1+1} + \sum (F'_{ij}H_1)M_{ij} \\ &\in \langle \{F_{ij}\} \rangle. \end{aligned}$$

Since $F'_{u0} = F_{10} = g$, $F'_{um_u+1} = F_{1m_1+1} = h$, $v(J) = \sum_{i=1}^{u} m_i + 3$ where the degrees of the minimal generators of J are

$$\begin{gathered} \beta - d_{1m_1} + \alpha - 1, \ \ldots, \ \beta - d_{11} + \alpha - m_1, \\ \beta - d_{2m_2} + \alpha, \ \ldots, \ \beta - d_{21} + \alpha - m_2 + 1, \\ \vdots \\ u, \ \alpha, \ \beta - d_{um_u} + \alpha + u - 2, \ \ldots, \ \beta - d_{u1} + \alpha - m_u + u - 1, \ \beta. \end{gathered}$$

Hence we are done. □

Remark 4.10. Let $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$ where $\mathbb{X}_i$ is a k-configuration in $\mathbb{P}^3$ of type $(d_{i1}, \ldots, d_{im_i})$ contained in the hyperplane $\mathbb{H}_i$. Assume that the hyperplanes $\mathbb{H}_i$ are parallel to each other. Since $\mathbb{X}_i$ is a k-configuration in $\mathbb{H}_i$ of type $(d_{i1}, \ldots, d_{im_i})$, there exist subsets $\mathbb{X}_{i1}, \ldots, \mathbb{X}_{im_i}$ and distinct lines $\mathbb{L}_{i1}, \ldots, \mathbb{L}_{im_i}$ which are contained in $\mathbb{H}_i$ such that:

(1) $\mathbb{X}_i = \bigcup_{k=1}^{m_i} \mathbb{X}_{ik}$;

(2) $|\mathbb{X}_{ik}| = d_{ik}$ and $\mathbb{X}_{ik} \subset \mathbb{L}_{ik}$ for each $k = 1, \ldots, m_i$.

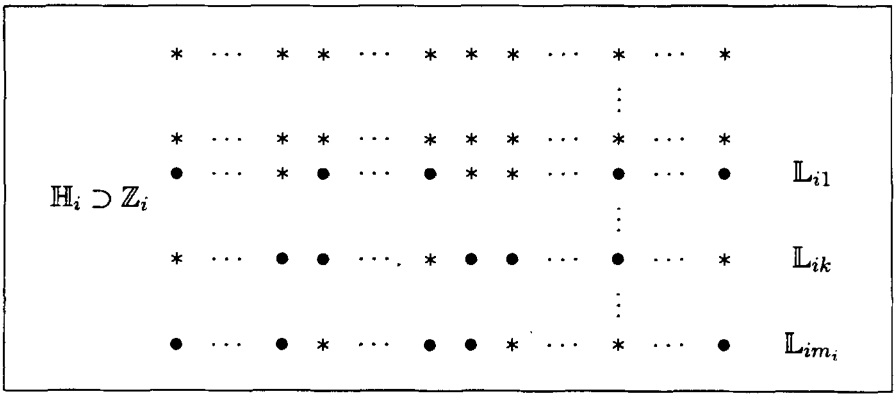


Fig. 2. $\mathbb{X}_i$ is the set of all •'s. $\mathbb{Y}_i$ is the set of all $*$'s.

Choose α and β such that $m_u < \alpha$, $d_{um_u} < \beta$, and $\alpha < \beta$. Let $\mathbb{Y}_i$ be the weak k-configuration in $\mathbb{P}^3$ of type $(\beta - d_{im_i}, \dots, \beta - d_{i1}, \beta, \dots, \beta)$ which is obtained by taking the complement of $\mathbb{X}_i$ in a set $\mathbb{Z}_i$ where $\mathbb{Z}_i$ is constructed as follows.

To each line $\mathbb{L}_{ik}$ of $\mathbb{H}_i$, add $\beta - d_{ik}$ distinct new points. Further add $\alpha - m_i$ new lines each containing β distinct points. (This set will then contain $\alpha\beta$ distinct points. See Fig. 2)

Let $\mathbb{Y} := \bigcup_{i=1}^{u} \mathbb{Y}_i$ and $J = I(\mathbb{Y})$. Then $\mathbb{Y}$ is a weak k-configuration in $\mathbb{P}^3$ of type $(\beta - d_{um_u}, \dots, \beta - d_{u1}, \beta, \dots, \beta; \dots; \beta - d_{1m_1}, \dots, \beta - d_{11}, \beta, \dots, \beta)$. From the proof of Theorem 4.9, we can see that $v(J) \le \sum_{i=1}^{u} m_i + 2u + 1$. The following example shows that each case of the above inequality can occur.

Example 4.11 (Macaulay, see Bayer and Stillman [1]). Consider the following examples.

(1) Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^3$ of type $(2,3,5)$ and $\mathbb{Y}_1 \subset \mathbb{Z}$ be a weak k-configuration in $\mathbb{P}^3$ of type $(3,4,5;4,5,5)$ (see Fig. 3). Then the number of minimal generators of the ideal of $\mathbb{Y}_1$ is 6 by Theorem 4.9.

(2) Let

$$\begin{aligned}\mathbb{Y}_2 = \{ &(1,2,1,1),\ (2,4,1,1),\ (3,6,1,1),\ (0,1,1,1),\ (1,3,1,1),\\ &(2,5,1,1),\ (3,7,1,1),\ (0,-1,1,1),\ (1,1,1,1),\ (2,3,1,1),\\ &(3,5,1,1),\ (0,1,0,1),\ (0,2,0,1),\ (0,3,0,1),\ (1,2,0,1),\\ &(1,3,0,1),\ (2,0,0,1),\ (2,1,0,1),\ (2,2,0,1),\ (2,3,0,1)\}.\end{aligned}$$

(See Fig. 4). Then $\mathbb{Y}_2$ is a weak k-configuration in $\mathbb{P}^3$ of type $(2,3,4;3,4,4)$, and the number of minimal generators of the ideal of $\mathbb{Y}_2$ is 7 from Macaulay [1].

(3) Let

$$\mathbb{Y}_3' = \{(4,8,1,1),\ (4,9,1,1),\ (4,7,1,1),\ (0,4,0,1),\ (1,4,0,1),\ (2,4,0,1)\}$$

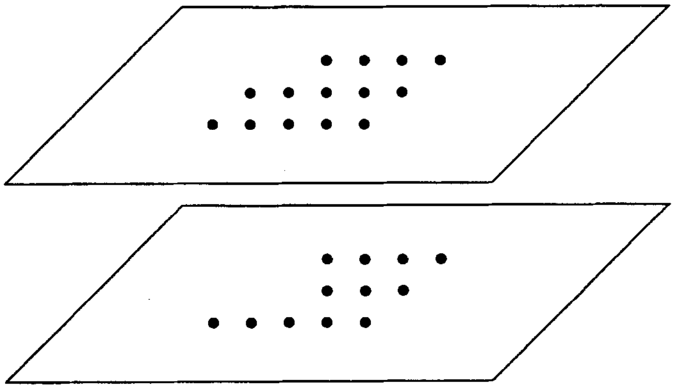


Fig. 3. A weak k-configuration in $\mathbb{P}^3$ of type $(3,4,5;4,5,5)$.

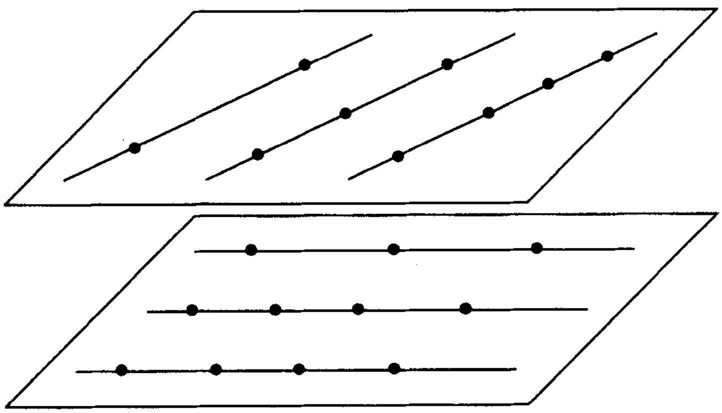


Fig. 4. A weak k-configuration in $\mathbb{P}^3$ of type $(2,3,4;3,4,4)$.

and $\mathbb{Y}_3 = \mathbb{Y}_2 \cup \mathbb{Y}_3'$ (see Fig. 5). Then $\mathbb{Y}_3$ is a weak k-configuration in $\mathbb{P}^3$ of type $(3,4,5;4,5,5)$, and the number of minimal generators of the ideal of $\mathbb{Y}_3$ is 8 from Macaulay [1].

Corollary 4.12. *Let* $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, *and* J *be as in Remark* 4.6 *and let* $I = I(\mathbb{X})$. *Then* $I+J$ *is a Gorenstein ideal of codimension* 4 *and*

$$v(I+J) = 2\sum_{i=1}^{u} m_i + u + 1.$$

Proof. By Remark 1.4 in [7], $I+J$ is a Gorenstein ideal of codimension 4. Let H_i be as in the proof of Theorem 4.9 and $H = \prod_{i=1}^{u} H_i$. Let $\{H, F_{10}, F_{11}, \ldots, F_{1m_1}; \ldots; F_{u0}, F_{u1}, \ldots, F_{um_u}\}$ be the set of the minimal generators of I and let $\{H, G_{10}, \ldots, G_{1m_1}, G_{1m_1+1}; G_{21}, \ldots, G_{2m_2}; \ldots; G_{u1}, \ldots, G_{um_u}\}$ be the set of the minimal generators of J where $F_{u0} | G_{10}$

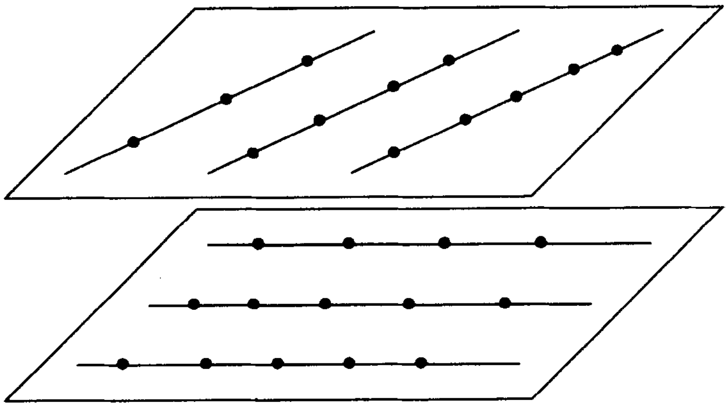


Fig. 5. A weak k-configuration in $\mathbb{P}^3$ of type (3, 4, 5; 4, 4, 5).

and $F_{um_u}|G_{1m_1+1}$. (This is always possible.) So we have that

$$H,\ F_{10},\ F_{11},\ \dots,\ F_{1m_1},\ \dots,\ F_{u0},\ F_{u1},\ \dots,\ F_{um_u},$$
$$G_{11},\ \dots,\ G_{1m_1},\ G_{1m_1+1},\ G_{21},\ \dots,\ G_{2m_2},\ \dots,\ G_{u1},\ \dots,\ G_{um_u}$$

certainly generate $I+J$.

We first show that no other F_{ij} can be eliminated from the set. If $F_{ij}\langle H,\dots,\widehat{F_{ij}},\dots,F_{um_u};G_{11},\dots,G_{um_u}\rangle$ (where $\hat{*}$ means that $*$ is omitted), then

$$F_{ij}=\alpha H+\alpha_{10}F_{10}+\cdots+\widehat{\alpha_{ij}F_{ij}}+\cdots+\alpha_{um_u}F_{um_u}+\beta_{11}G_{11}+\cdots+\beta_{um_u}G_{um_u}$$

for some $\alpha,\alpha_{10},\dots,\widehat{\alpha}_{ij},\dots,\alpha_{um_u},\beta_{11},\dots,\beta_{um_u}\in R$. Thus

$$\begin{aligned}\alpha_{10}F_{10}+\cdots-F_{ij}+\cdots+\alpha_{um_u}F_{um_u}&=-(\alpha H+\beta_{11}G_{11}+\cdots+\beta_{um_u}G_{um_u})\\&\in I\cap J=\langle H,G_{10},G_{1m_1+1}\rangle.\end{aligned}$$

Hence there exist $\alpha',\alpha'',\alpha'''\in R$ such that

$$\alpha_{10}F_{10}+\cdots-F_{ij}+\cdots+\alpha_{um_u}F_{um_u}=-(\alpha'H+\alpha''G_{10}+\alpha'''G_{1m_1+1}),$$

i.e.,

$$\begin{aligned}F_{ij}&=\alpha'H+\alpha_{10}F_{10}+\cdots+\widehat{\alpha_{ij}F_{ij}}+\cdots+\alpha_{um_u}F_{um_u}+\alpha''G_{10}+\alpha'''G_{1m_1+1}\\&\in\langle H,F_{10},\dots,\widehat{F_{ij}},\dots,F_{um_u}\rangle,\end{aligned}$$

a contradiction. Hence $F_{ij}\notin\langle H,F_{10},\dots,\widehat{F_{ij}},\dots,F_{um_u},G_{11},\dots,G_{um_u}\rangle$.

We now show that no G_{kl} can be eliminated from this set. Assume $G_{kl}\in\langle H,F_{10},\dots,F_{um_u},G_{11},\dots,\widehat{G_{kl}},\dots,G_{um_u}\rangle$. Then

$$G_{kl}=\alpha H+\alpha_{10}F_{10}+\cdots+\alpha_{um_u}F_{um_u}+\beta_{11}G_{11}+\cdots+\widehat{\beta_{kl}G_{kl}}+\cdots+\beta_{um_u}G_{um_u}$$

for some $\alpha, \alpha_{10}, \ldots, \alpha_{um_u}, \beta_{11}, \ldots, \widehat{\beta}_{kl}, \ldots, \beta_{um_u} \in R$. Thus

$$\begin{aligned}-(\alpha H + \alpha_{10}F_{10} + \cdots + \alpha_{um_u}F_{um_u}) &= \beta_{11}G_{11} + \cdots - G_{kl} + \cdots + \beta_{um_u}G_{um_u} \\ &\in I \cap J = \langle H, G_{10}, G_{1m_1+1}\rangle.\end{aligned}$$

Hence

$$\beta_{11}G_{11} + \cdots - G_{kl} + \cdots + \beta_{um_u}G_{um_u} = -(\beta H + \beta' G_{10} + \beta'' G_{1m_1+1})$$

for some $\beta, \beta', \beta'' \in R$. It follows that

$$\begin{aligned}G_{kl} &= \beta H + \beta' G_{10} + \beta_{11}G_{11} + \cdots + \widehat{\beta_{kl}G_{kl}} + \cdots + \beta_{um_u}G_{um_u} + \beta'' G_{1m_1+1} \\ &\in \langle H, G_{10}, \ldots, \widehat{G}_{kl}, \ldots, G_{um_u}\rangle,\end{aligned}$$

a contradiction. Thus $G_{kl} \notin \langle H, F_{10}, \ldots, F_{um_u}, G_{11}, \ldots, \widehat{G}_{kl}, \ldots, G_{um_u}\rangle$, we are done. □

Remark 4.13. Let $\mathbb{Z} = \bigcup_{i=1}^{u} \mathbb{Z}_i$ be a basic configuration in $\mathbb{P}^3$ of type (u, m_u, d_{um_u}) $(u \geq 2)$ where $\mathbb{Z}_i$ is a basic configuration in $\mathbb{P}^3$ of type $(1, m_u, d_{um_u})$ and $\mathbb{X} \subset \mathbb{Z}$ be a k-configuration in $\mathbb{P}^3$ of type $(d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$. Let $\mathbb{X}_i = \mathbb{Z}_i \cap \mathbb{X}$. Then $\mathbb{X} = \bigcup_{i=1}^{u} \mathbb{X}_i$. Let $\mathbb{Y}_i = \mathbb{Z}_i - \mathbb{X}_i$. Then $\mathbb{Y} =: \mathbb{Z} - \mathbb{X} = \bigcup_{i=1}^{u} \mathbb{Y}_i$ and $\mathbb{Y}$ is a weak k-configuration in $\mathbb{P}^3$. Assume that $\mathbb{Y}_i$ is a weak k-configuration in $\mathbb{P}^2$ of type $(d_{um_u} - d_{im_i}, \ldots, d_{um_u} - d_{i1}, d_{um_u}, \ldots, d_{um_u})$.

The proof of the following theorem is the same as that of Theorem 4.9, so we shall omit it.

Theorem 4.14. *Let $\mathbb{Y}$ be as in Remark 4.13. Let $J = I(\mathbb{Y})$. Then $\nu(J) = \sum_{i=1}^{u} m_i + 3$ and the degrees of the minimal generators of J are*:

$$\begin{aligned}&u,\ d_{um_u} - d_{1m_1} + m_u - 1,\ \ldots,\ d_{um_u} - d_{11} + m_u - m_1,\\ &\qquad\vdots\\ &m_u,\ d_{um_u} - d_{u-1m_{u-1}} + m_u + u - 3,\ \ldots,\\ &d_{um_u} - d_{u-11} + m_u - m_{u-1} + u - 2,\ d_{um_u},\\ &m_u + u - 2,\ d_{um_u} - d_{um_u-1} + m_u + u - 3,\ \ldots,\ d_{um_u} - d_{u1} + u - 1.\end{aligned}$$

We also get the following corollary by the same method as in the proof of Corollary 4.12.

Corollary 4.15. *Let $\mathbb{X}$ and $\mathbb{Y}$ be as in Remark 4.13. Let $I = I(\mathbb{X})$ and $J = I(\mathbb{Y})$. Then $I + J$ is a Gorenstein ideal of codimension 4 and*

$$\nu(I + J) = 2\sum_{i=1}^{u} m_i + u + 1.$$

Acknowledgements

The author is grateful to Professor Tony Geramita for their interesting and motivating discussions throughout the course of this work and would like to thank the Mathematics and Statistics Department of Queen's University, Kingston, Ontario, for their kind hospitality during the preparation of this work.

References

[1] D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra, Source and object code available for Unix and Macintosh computers. Contact the authors, or download from zariski.harvard.edu via anonymous ftp.
[2] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977) 447–485.
[3] A.V. Geramita, P. Maroscia and L. Roberts, The Hilbert function of a reduced K-algebra, J. London Math. Soc. (2) 28 (1983) 443–452.
[4] A.V. Geramita and J.C. Migliore, Reduced Gorenstein codimension three subschemes of projective space, Proc. Amer. Math. Soc., to appear.
[5] A.V. Geramita, M. Pucci and Y.S. Shin, The smooth points in Gor(T), Queen's in Pure and Applied Math., The Curves Seminar at Queen's, Vol. X, 102 (1996) 256–297.
[6] T. Harima, Some examples of unimodal Gorenstein sequences, Preprint.
[7] C. Peskine et L. Szpiro, Liasion des variétés algébriques I, Invent. Math. 26 (1974) 271–302.
[8] L. Robert and M. Roitman, On Hilbert functions of reduced and of integral algebra, J. Pure Appl. Algebra 56 (1989) 85–104.